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WAVE PROPAGATION AND RESONANCE PHENOMENA IN INHOMOGENEOUS MEDIA

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The waveguide and resonance properties of inhomogeneous penetrable one-dimensional-periodical structures that consist of two different media are studied within the framework of a one-dimensional approximation. The pass and stop bands are determined. A dispersion relation for all the waveguide modes is obtained. Explicit expressions for low waveguide frequencies and corresponding phase velocities of waveguide modes for mono- and polydisperse media are found. The influence of the polydispersity of the sizes of heterogeneities on the low frequencies of a pass band is considered. A pass band in the range of low frequencies is detected. It is shown that the polydispersity does not affect the waveguide properties of a medium at low frequencies of the first pass band. The resonance phenomena in periodical media and structures are investigated. The resonance phenomena are shown to occur for an unlimited discrete set of frequencies if the group velocity of the waveguide mode for them is zero; in this case, the growth of the oscillation amplitude is localized in the neighborhood of a source (localization of the resonance). A synchrophasotron resonance at which the infinite chain of oscillation sources has the oscillations phase of a corresponding traveling wave from the pass band is detected.

Introduction. The study of wave propagation in inhomogeneous one-dimensional media is of significance for the solution of applied problems. The examples of such media upon propagation of acoustic waves in them are liquids with gas bubbles, composites, inhomogeneous mixtures with the periodical inclusion of components, foams, and porous and granular structures. For applied problems, of importance is to study the decelerating properties of inhomogeneous one-dimensional-periodical structures, determine the pass and stop bands, and study the resonance properties of periodical inhomogeneous structures with compact and distributed oscillation sources.

The first results and the bibliography are given by Brillouin and Parodi [1] and Sanchez-Palensia [2]. In the present work which is an extension and generalization of [3–5], the waveguide, decelerating, and resonance properties of inhomogeneous one-dimensional-periodical penetrable media of the type of a chain of gas bubbles in a liquid or a structured composite are studied within the framework of the one-dimensional theory. Within the framework of the two-dimensional theory, the wave propagation near the one-dimensional-periodical chains of penetrable and impenetrable obstacles was investigated in [6].

The studies performed in the present work can serve as a basis for the development of acoustic filters and decelerating media and the optimization of damping materials. The studies of resonance phenomena can be used, for example, for calculation of the chain of explosions for particle acceleration or production of absorbing materials.

1. Formulation of the Problems and Their Properties of Symmetry. The direct methods of studying the wave propagation in inhomogeneous periodical structures cannot be implemented because of a large number of heterogeneities. In this connection, the study of the fine structure of the frequency spectrum of a problem that describes steady-state oscillations in inhomogeneous one-dimensional-periodical media becomes important.

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Let an inhomogeneous one-dimensional-periodical medium consist of two components $M_1 = \{c_1, \rho_1\}$ and $M_2 = \{c_2, \rho_2\}$, where c_i and ρ_i is the velocity of sound and the density in the quiescent state ($i = 1, 2$) and $p^{(1)}$ and $p^{(2)}$ are the acoustic pressure perturbations in the first and second media, respectively. The medium M_1 is assumed to be denser than the medium M_2. The examples of the media in which $\tau \ll \infty$ ($\tau = \rho_2/\rho_1$ is the density ratio) are a liquid and a gas. One can consider that a chain of bubbles of the medium M_1 is placed in the medium M_2 or a chain of drops of the medium M_2 is placed in the medium M_1. It is assumed that the chain is spatially periodical. The following notation is used: $æ = c_1/c_2$ ia the ratio between the velocities of sound, ω is the circular frequency of oscillations, $\lambda = \omega L/c_1$ is the dimensionless oscillation frequency, L is the smallest spatial period of the one-dimensional-periodical media, and $\tilde{x} = x/L$ is the dimensionless spatial variable (hereafter, the tilde above x is omitted). In these variables, the smallest spatial period of an inhomogeneous medium is equal to unity. The subscripts 1 and 2 correspond to the media M_1 and M_2, respectively. The part of the medium whose length is equal to unity is called the fundamental cell.

Equations and Boundary Conditions. The steady-state acoustic pressure oscillations with circular frequency ω in the media M_1 and M_2 are described by the equations

$$p^{(1)}_{xx} + \lambda^2 p^{(1)} = 0, \qquad p^{(2)}_{xx} + \lambda^2 æ^2 p^{(2)} = 0. \tag{1.1}$$

The following conditions of pressure and velocity continuity (dynamic and kinematic conditions) should be fulfilled at the boundaries of contact of the media:

$$p^{(1)} = p^{(2)}, \qquad \tau p^{(1)}_x = p^{(2)}_x. \tag{1.2}$$

Hereinafter, relations (1.1) and (1.2) are called the problem T which completely describes the propagation of acoustic waves in inhomogeneous one-dimensional-periodical media.

Taking into Account the Interaction of Adjacent Heterogeneities. Let the medium M_2 (gas) of extent D (a single bubble) be in the medium M_1 (liquid) and the coordinate origin be chosen at the center of M_2. Within the framework of the one-dimensional theory, the free acoustic oscillations of a single gas bubble in a liquid are described by Eqs. (1.1) and the boundary conditions (1.2). In addition, the external medium should be subject to radiation conditions which have the following form for an acoustic pressure perturbation:

$$p^{(1)} = a_1 \exp(i\lambda x) \quad (x > D/2), \qquad p^{(2)} = a_2 \exp(-i\lambda x) \quad (x < -D/2). \tag{1.3}$$

The free oscillations of the bubble are determined by a set of complex numbers λ^*_k ($k = 1, 2, \ldots$) for which nontrivial solutions of the problem $B = \{(1.1), (1.2), (1.3)\}$ exist. These numbers are called the quasi-eigenfrequencies of the problem B, and the corresponding oscillations are called the quasi-eigenoscillations. One can show that

$$\lambda^*_k = k\pi/(æD) + i \ln[(æ + \tau)/(æ - \tau)]/(æD) \qquad (k = 1, 2, \ldots). \tag{1.4}$$

It is noteworthy that the quasi-eigenfrequencies of the problem B are a continuous function of τ ($0 < \tau < \infty$). The physical meaning of the real and imaginary parts of the quasi-eigenfrequencies is clear [2]: the quantity $\mathrm{Re}(\lambda^*_k)$ is the dimensionless frequency of quasi-eigenoscillations, and the damping in time at a fixed point of space is determined by the expression $p(x, t) = A(x) \exp[-c_1 \mathrm{Im}(\lambda^*_k)t]$. As $\tau \to 0$, the quasi-eigenfrequencies tend to the eigenfrequencies of the Neumann problem in the domain $-D/2 < x < D/2$. The quasi-eigenvalues of a Helmholtz resonator for a small parameter (radius of the throat) have a similar property [2]. For this resonator, the quasi-eigenvalues and the corresponding oscillations with a wavelength much larger than its geometrical dimensions exist, i.e., the so-called Helmholtz mode [2]. The quasi-eigenvalue of $\lambda^*_0(\tau)$ such that $\lim_{\tau\to 0} \lambda^*_0(\tau) = 0$ and $\mathrm{Re}[\lambda^*_0(\tau)] \neq 0$ corresponds to this mode; the latter means the existence of eigenoscillations. It follows from (1.4) that $\lambda^*_0 = i \ln[(æ+\tau)/(æ-\tau)]/(æD)$ and $\mathrm{Re}(\lambda^*_0) = 0$, i.e., there are no eigenoscillations of the Helmholtz mode for a single bubble.

A structure that consists of two gas bubbles in a liquid possesses a different property. The quasi-eigenfrequency of oscillations λ are determined from the equation

$$(æ + \tau)^2 \exp[-(1/2)i\lambda(H + 2æL)] + (\tau^2 - æ^2) \exp[(1/2)i\lambda(H + 2æL)]$$
$$= (\tau^2 - æ^2) \exp[-(1/2)i\lambda(-H + 2æL)] + (æ - \tau)^2 \exp[(1/2)i\lambda(-H + 2æL)]. \tag{1.5}$$

If $\tau = 0$, the quasi-eigenfrequencies become the eigenfrequencies of oscillations and they are determined by the equality $\cos(\lambda H/2) \sin(\lambda æL) = 0$. As a result of the interaction between the bubbles, oscillations in the Helmholtz mode appear. The closest-to-zero quasi-eigenfrequency λ^*_0 of an ensemble that consists of two bubbles is calculated with the use of Eq. (1.5):

$$\lambda_0^* = \pm\tau \frac{\sqrt{\tau^2 H^2 + 4\tau^2 æ^2 D^2 + 4\tau^3 HD}}{\tau^2 H^2 + 2\tau^2 æ^2 D^2 + 2\tau^3 HD + 2æ^4 L^2 + 2æ^2 \tau HL}$$
$$- i\,\frac{\tau(H\tau + 2æ^2 L)}{\tau^2 H^2 + 2\tau^2 æ^2 D^2 + 2\tau^3 HD + 2æ^4 L^2 + 2æ^2 \tau HL}. \tag{1.6}$$

Here D is the bubble diameter, H is the distance between the bubbles, and i is the imaginary unity. It is noteworthy that the frequency characteristic of the oscillations of the ensemble from two bubbles (1.6) differs greatly from the characteristic of the oscillations of one bubble (1.4). Making allowance for the interaction between the bubbles gives rise to the appearance of low-frequency eigenoscillations.

It follows from the above example that the usual averaging methods can cause the loss of solutions or the distortion of the qualitative properties of the structure behavior if the interaction of the entire ensemble of heterogeneities is not taken into account. Hereafter, the interaction of all the heterogeneities in the one-dimensional-periodical chain is taken into account with the use of the representations of admissible symmetry groups in the space of possible solutions of the problem.

Symmetry Properties. Because the wave equation is invariant with respect to any locally plane symmetries, the symmetry of the problem T is determined by the symmetry of the chain of heterogeneities. All the chains of heterogeneities can be classified according to the groups of admissible symmetries. Using the methods of the theory of symmetry groups [7, 8], one can show that only two types of one-dimensional-periodical chains of heterogeneities are possible. Type I: a chain of heterogeneities that admits only the translation group $\{T_1\}$ [7], where $T_1\langle u(x,y)\rangle = u(x, y+1)$, $T_n = (T_1)^n$. Type II: a chain that admits only the symmetry group $\{T_1, D_1^x\}$, where $D_1^x\langle u(x,y)\rangle = u(x,-y)$. Figure 1 shows one of the possible one-dimensional-periodical chains of type II, and Fig. 6 shows that of type I. One-dimensional-periodical chains that admit another symmetry groups do not exist [7, 8].

The symmetry group S_T of the problem T allows us to decompose the space of admissible solutions of this problem into subspaces invariant with respect to the representations of S_T in the space of solutions. By definition, all the one-dimensional-periodical structures admit the group $\{T_1\}$; therefore, the space of admissible solutions can be decompose into subspaces invariant with respect to this group [9]. For the function $p(x)$ which belongs to such subspaces, a value of ξ $(-\pi \leqslant \xi \leqslant \pi)$ such that $T_1\langle p(x)\rangle \equiv p(x+1) \equiv \exp(i\xi)p(x)$ exists. Therefore,

$$p(x) \equiv a(x)\exp(i\xi x), \qquad a(x+1) \equiv a(x). \tag{1.7}$$

Here i is the imaginary unity and ξ is the phase shift of oscillations in the adjacent fundamental domains of the translation group. Hereafter, the problem T with condition (1.7) is called the problem $T(\xi)$.

For heterogeneity chains of type II, four one-dimensional nonreducible representations of a symmetry group $S_T = \{T_1, D_1^x\}$ in the space of admissible solutions [9] are possible:

$$\begin{gathered} \{\tau_1(T_1) = -1,\ \tau_1(D_1^x) = +1\}, \qquad \{\tau_2(T_1) = -1,\ \tau_2(D_1^x) = -1\}, \\ \{\tau_3(T_1) = +1,\ \tau_3(D_1^x) = +1\}, \qquad \{\tau_4(T_1) = +1,\ \tau_4(D_1^x) = -1\}. \end{gathered} \tag{1.8}$$

It suffices to study the problem $T(\xi)$ in a certain fundamental cell of a translation group (a certain period of the structure), for example, in the interval $0 < x < 1$. The solution on the entire straight line can be obtained by continuation of the solution of the problem in one period with the use of (1.7).

Waveguide Modes and In-Phase Oscillations. The traveling waves that propagate in a chain of heterogeneities without damping correspond to the waveguide modes of oscillations.

Definition 1. The nontrivial solution of the problem $T(\xi)$ for $\xi \neq 0$ is called the waveguide function. The corresponding value of the parameter λ^* is called the dimensionless waveguide frequency (waveguide eigenvalue), and $\omega^* = \lambda^* c_1/L$ is called the cyclic waveguide frequency of the problem $T(\xi)$.

If $\xi = 0$, the oscillations are in-phase in any adjacent fundamental cells of a translation group. The form of these oscillations is described by the problem T for functions invariant with respect to the representations τ_3 and τ_4 from (1.8).

By virtue of Eq. (1.7), any waveguide function has the form $p(x) \equiv a(x)\exp(i\xi x)$. The quantity ξ is the wave number of the waveguide mode, the amplitude of the waveguide mode $a(x+1) \equiv a(x)$ is a space-periodical complex-valued function, λ_k^* is the dimensionless waveguide frequency, and the relations $\lambda_k^* = \lambda_k^*(\xi)$, where $k = 1, 2, \ldots, K$, are the dispersion relations for waveguide modes. The semienclosed intervals $\sigma_n = \left(\inf\limits_{0\leqslant\xi\leqslant\pi}[\lambda_n(\xi)], \sup\limits_{0\leqslant\xi\leqslant\pi}[\lambda_n(\xi)]\right]$ $(n = 1, 2, \ldots)$ of dimensionless frequencies are the pass bands for the waveguide modes $\lambda_k = \lambda_k(\xi)$ $(k = 1, 2, \ldots, K)$.

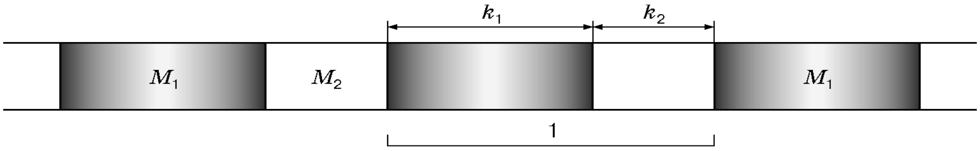

Fig. 1. Monodisperse chain of bubbles.

To verify the correctness of the problem $T(\xi)$ and numerical studies, we need the following statement.

Statement 1. *The set of waveguide frequencies of the problem $T(\xi)$ is discrete on a real axis.*

Proof. The discreteness of waveguide frequencies in the topology of a real axis follows from the analyticity of the resolvent of the problem $T(\xi)$. If one multiplies (1.1) by the complex conjugate functions p_1 and p_2, respectively, and integrate by parts over the fundamental cell of a translation group, one can obtain the relation

$$\int\limits_{M_1} \tau[\,|p_x^{(1)}|^2 - \lambda^2|p^{(1)}|^2] + \int\limits_{M_2} [\,|p_x^{(2)}|^2 - \lambda^2 æ^2|p^{(2)}|^2] = 0,$$

from which follows the real-valued character of the waveguide frequencies of the problem $T(\xi)$. Statement 1 is proved.

2. Waveguide Properties of a Monodisperse Chain of Heterogeneities. For a one-dimensional-periodical chain of identical heterogeneities of type II (monodisperse chain) (Fig. 1), the problem $T(\xi)$ is the simplest problem. Let the length of one connected layer of the medium M_1 (linear concentration) be equal to k_1, and the length of one connected layer of the medium M_2 (linear concentration) be equal to k_2. Since $1 = k_1 + k_2$ is the dimensionless spatial period of the structure (Fig. 1), we have $k_2 = 1 - k_1$, and it suffices to use k_1 (hereinafter, $k_1 = k$) to describe completely the monodisperse chain of heterogeneities. The oscillations in the fundamental cell $0 \leqslant x \leqslant 1$ are described by Eqs. (1.1), the transmission conditions (1.2), and the conditions of the phase shift of oscillations in the adjacent fundamental cells (1.7). The conditions at the boundaries of the fundamental cell

$$p^{(1)}(-k/2)\exp(i\xi) = p^{(2)}(1-k/2), \qquad \tau p_x^{(1)}(-k/2)\exp(i\xi) = p_x^{(2)}(1-k/2) \tag{2.1}$$

are equivalent to (1.7).

Hereinafter, Eqs. (1.1), (1.2), and (1.7) are called the problem $TM(\xi)$. It is noteworthy that the family of problems $TM(\xi)$ completely takes into account the possible interactions of all the heterogeneities in a one-dimensional-periodical chain.

Dispersion Relations. In the domains $\{x\colon -k/2 \leqslant x \leqslant k/2\}$ and $\{x\colon k/2 \leqslant x \leqslant 1-k/2\}$ occupied by the media M_1 and M_2, the general solution of Eq. (1.1) has the form $p_1 = a_1 \exp(i\lambda x) + b_1 \exp(-i\lambda x)$ and $p_2 = a_2 \exp(i\lambda æ x) + b_2 \exp(-i\lambda æ x)$. Therefore, the problem $TM(\xi)$ is equivalent to the linear system of equations $A(\lambda)Y = 0$ with desired $(a_1, b_1, a_2, b_2) = Y$. The matrix $A(\lambda)$ of this system has the form

$$A(\lambda) = \begin{pmatrix} \exp\Big(i\,\dfrac{\lambda k}{2}\Big) & \exp\Big(-i\,\dfrac{\lambda k}{2}\Big) & -\exp\Big(i\,\dfrac{\lambda k æ}{2}\Big) & -\exp\Big(-i\,\dfrac{\lambda k æ}{2}\Big) \\ \tau\exp\Big(i\,\dfrac{\lambda k}{2}\Big) & -\tau\exp\Big(-i\,\dfrac{\lambda k}{2}\Big) & -æ\exp\Big(i\,\dfrac{\lambda k æ}{2}\Big) & æ\exp\Big(-i\,\dfrac{\lambda k æ}{2}\Big) \\ \exp\Big[i\Big(-\dfrac{\lambda k}{2}+\xi\Big)\Big] & \exp\Big[i\Big(\dfrac{\lambda k}{2}+\xi\Big)\Big] & -\exp\Big[i\lambda æ\Big(1-\dfrac{k}{2}\Big)\Big] & -\exp\Big[-i\lambda æ\Big(1-\dfrac{k}{2}\Big)\Big] \\ \tau\exp\Big[i\Big(-\dfrac{\lambda k}{2}+\xi\Big)\Big] & -\tau\exp\Big[i\Big(\dfrac{\lambda k}{2}+\xi\Big)\Big] & -æ\exp\Big[i\lambda æ\Big(1-\dfrac{k}{2}\Big)\Big] & æ\exp\Big[-i\lambda æ\Big(1-\dfrac{k}{2}\Big)\Big] \end{pmatrix}.$$

A nontrivial solution of the problem $TM(\xi)$ exists if the determinant of matrix $A(\lambda)$ is equal to zero. Therefore, the waveguide frequencies of problem $TM(\xi)$ are the zeros of the analytical function $\det[A(\lambda)]$. Whence, the waveguide frequencies $\lambda^*(\tau, \xi)$ of the problem $TM(\xi)$ are discrete on a real axis and they are a continuous function of τ (on the set $0 \leqslant \tau < 1$) and ξ ($|\xi| \leqslant \pi$). For fixed æ, τ, and k, the equation $\det[A(\lambda)] = 0$ is dispersion relations for all the waveguide modes $\lambda_n = \lambda_n(\xi)$ ($n = 1, 2, \ldots$), which are connected components of the set of all waveguide frequencies of the problem $TM(\xi)$ on the plane (ξ, λ).

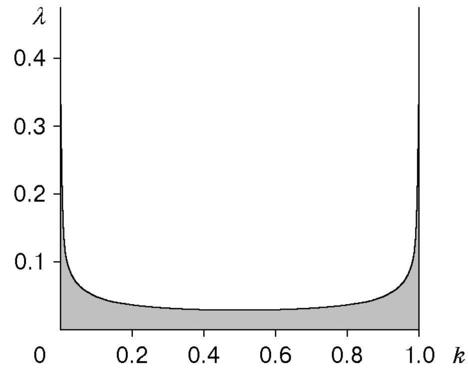

Fig. 2

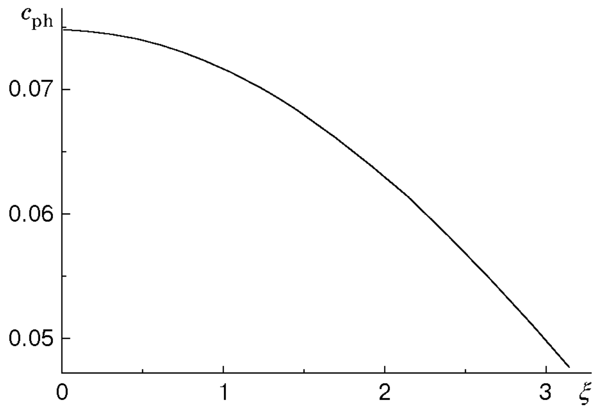

Fig. 3

Fig. 2. The pass-band width of the creeping mode versus concentration.

Fig. 3. Phase velocities of the creeping mode versus the wave number.

For all the waveguide modes of the problem $TM(\xi)$, the dispersion relations have the form

$$4\tau æ[1+\cos(2\xi)] - (\tau+æ)^2\{\cos[\lambda(k-æk+æ)+\xi] + \cos[\lambda(k-æk+æ)-\xi]\}$$
$$+ (\tau-æ)^2\{\cos[\lambda(k+æk-æ)+\xi] + \cos[\lambda(k+æk-æ)-\xi]\} = 0. \tag{2.2}$$

The pass $\{\sigma_n\}_{n=1,2,\ldots}$ and stop bands are completely described by (2.2).

Creeping Mode. The study of the propagation of long (low-frequency) waves in a one-dimensional-periodical chain of heterogeneities is most important for various applications. In this case, the wavelength considerably exceeds the structure period and the heterogeneity sizes. The waveguide frequencies of the problem $TM(\xi)$ are a continuous function of τ, and $\lambda = 0$ is the solution of Eq. (2.2) for $\tau = 0$. It follows that a waveguide frequency $\lambda_1^*(\tau)$ of the problem $TM(\xi)$ such that $\lim\limits_{\tau\to 0}\lambda_1^*(\tau) = 0$ exists. This value corresponds to the lowest frequency of the waveguide oscillations of a monodisperse chain of bubbles. Hereafter, an oscillation mode that corresponds to the lowest waveguide frequency is called the creeping mode. It is noteworthy that the wavelength of the creeping mode exceeds the heterogeneity sizes. If one expands the determinant of matrix $A(\lambda)$ or dispersion relations (2.2) into a Taylor series in terms of λ at the point $\lambda = 0$ and ignores terms of the order λ^3, one can obtain an approximate expression for the low waveguide frequencies of the creeping mode

$$\lambda_1(\tau,\xi) = \sqrt{2\tau(1-\cos\xi)}\Big/\sqrt{(k+\tau-k\tau)(k\tau-kæ^2+æ^2)}. \tag{2.3}$$

For small τ, the representation $\lambda_1(\tau,\xi) = \sqrt{2\tau(1-\cos\xi)}\Big/\sqrt{æ^2k(1-k)}$ is true. For water and air, we have $æ = 1400/330$ and $\tau = 0.001$.

Figure 2 shows the width of the pass band of the waveguide frequency of the creeping mode versus the water concentration for fixed values of the parameters of two media (water and air). It is necessary to note that the waveguide frequency of the creeping mode depend strongly on the linear concentration k; for $k \approx 0$ and $k \approx 1$, the pass band for the creeping mode extends unboundedly. For various applications in which $\tau \ll 1$, it is important that there is a global minimum of the waveguide frequency as a function of linear concentration for $k = 0.5$. By virtue of (2.3), the following statement is true.

Statement 2. The smallest value of the waveguide frequency of the creeping mode as a function of linear concentration k is reached at the point $k = 1/2$.

Condition (1.7) allows one to present any solution of the problem $TM(\xi)$ as $p(x) = a(x)\exp(i\xi x)$, where $a(x) \equiv a(x+1)$ is the amplitude of waveguide oscillations. This representation makes it possible to consider ξ as a wave number. The dependence of the phase velocity $c_{\rm ph}^{(1)}(\xi,k,\tau)$ of the waveguide mode of a monodisperse chain of bubbles of the creeping mode on the wave number ξ is shown in Fig. 3. It is noteworthy that the phase velocity of waveguide modes in a monodisperse chain of bubbles can be smaller than the velocity of sound in the bubbles and the ambient medium.

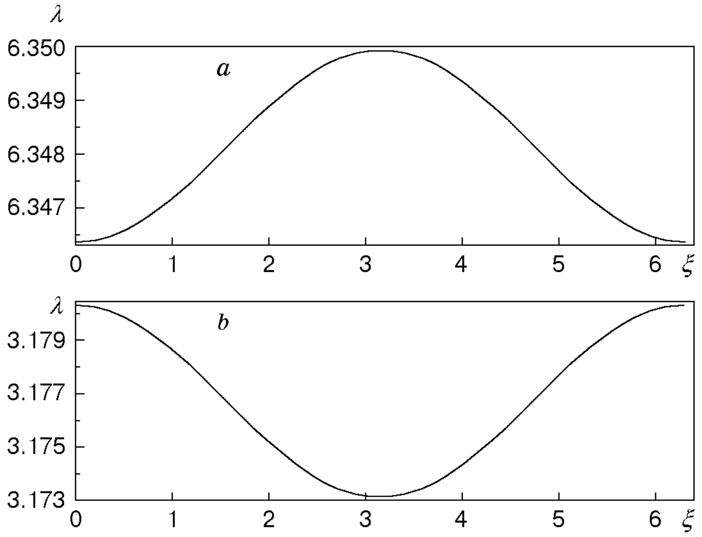

Fig. 4

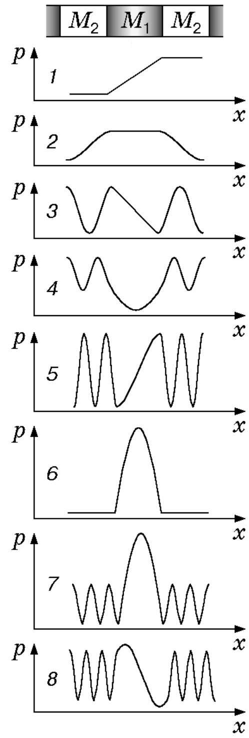

Fig. 5

Fig. 4. Dispersion curves of the second (a) and third (b) waveguide modes ($k = 0.99$).

Fig. 5. Pressure field for various waveguide modes.

Long-Wave Approximation. For various applications, it is expedient to consider the asymptotic behavior of the waveguide frequencies and phase velocities of the creeping mode provided that the wavelength is considerably larger than the spatial period of a chain of bubbles. For the creeping waveguide mode corresponding to the waveguide frequency $\lambda_1(\xi, k, \tau)$, the wavelength $L_w = 2\pi/\xi$. For large values of L_w, the wavenumber of the waveguide mode is close to zero ($\xi \approx 0$). For small ξ [see (2.3)], we have $\lambda_1(\xi, k, \tau) = \xi\sqrt{\tau}\Big/\sqrt{(k + \tau - k\tau)(k\tau - k\ae^2 + \ae^2)}$. The dimensionless phase velocity $c_{\rm ph}^{(1)}(\xi, k, \tau)$ of the creeping mode is determined as $c_{\rm ph}^{(1)}(\xi, k, \tau) = \lambda_1(\xi, k, \tau)/\xi = \sqrt{\tau}/\Big[\ae\sqrt{k(1-k)}\,\Big] = (c_2/c_1)\sqrt{\tau/k(1-k)}$.

It is noteworthy that the phase velocity of a long wave of the creeping mode depends only on the concentration and the ratio between the velocities of sound and the ratio between the densities of two media forming the chain.

Higher Waveguide Modes. Alongside with the creeping mode, higher-frequency waveguide modes can propagate in the chain of bubbles. The number of these modes is infinite. For small values of the parameter τ, it is possible to consider that the corresponding waveguide frequencies are localized in the vicinity of the corresponding eigenvalues of the Dirichlet problems in domain No. 1 and the Neumann problems in domain No. 2 for a one-dimensional Laplace operator with allowance for the velocities of sound.

Figure 4 shows dispersion curves for the second and third waveguide modes (the creeping mode is considered to be the first mode) in a chain consisting of water and air. Because the dimensionless length of an air bubble is equal to 0.01, the first waveguide frequencies are close to the eigenvalues of the Dirichlet problem for the Laplace operator in the interval [0, 0.99], which are, in turn, close to the numbers $n\pi$ ($n = 0, 1, 2, \ldots$). This is supported by calculations as well. The intervals of variation of the dispersion curves on the ordinate axis shown in Fig. 4 determine the pass bands of the chain.

Mechanics of Waveguide Oscillations. Figure 5 shows the form of waveguide modes (acoustic-pressure field) for the values of the parameters of the water–air chain ($\xi = \pi$ and $k = k_1 = 0.5$) and the values of the

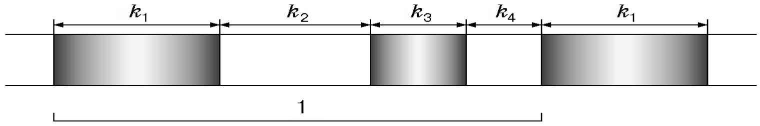

Fig. 6. Polydisperse chain of bubbles.

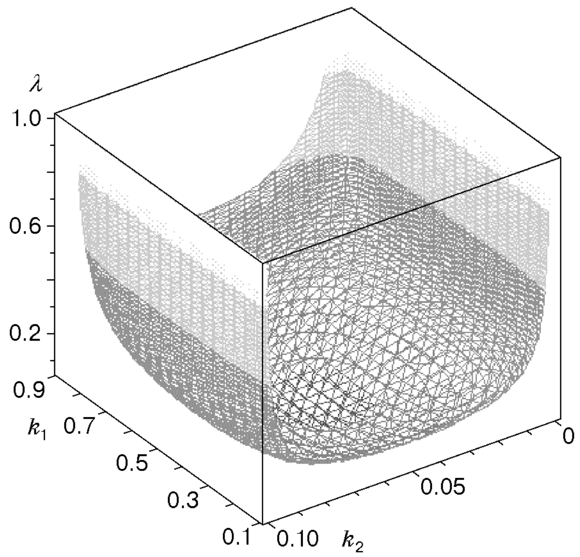

Fig. 7. Frequencies of the second waveguide modes versus the bubble sizes in a polydisperse chain.

dimensionless waveguide frequencies $\lambda_1 = 0.02981051656$, $\lambda_2 = 1.480950276$, $\lambda_3 = 2.962316191$, $\lambda_4 = 4.442661187$, $\lambda_5 = 5.924166151$, $\lambda_6 = 6.282808282$, $\lambda_7 = 7.4059953558$, and $\lambda_8 = 8.886050022$ (curves 1–8, respectively). The form of waveguide modes (Fig. 5) shows the mechanics of oscillations. For example, the oscillations are localized in water for λ_6 and air for λ_2. Upon oscillations in the first mode, the water drops move as a whole, and the air bubbles act as springs.

In real media, monodisperse structures occur rarely; therefore, it is worthwhile studying the propagation of acoustic oscillations in polydisperse chains of heterogeneities. The simplest example is a periodical chain with two different bubbles in the period.

3. Waveguide Properties of a Polydisperse Chain of Heterogeneities. Let two inclusions of the medium M_1 (water) with sizes k_1 and k_3 ($k_1 + k_3 = k$) and two inclusions of the medium M_2 (air bubbles) with sizes k_2 and k_4 be contained in one period of the chain (Fig. 6). The waveguide modes of the chain are described in one spatial period by relations (1.1) and (1.2) and the phase-shift conditions (1.7) or (2.1), which are equivalent to the system of equations for eight unknowns $AP(\lambda)Y = 0$. The matrix $AP(\lambda)$ is constructed in the same manner as for a monodisperse chain with appropriate changes. It follows from Statement 1 that the waveguide frequencies of oscillations of the chain are real. The lowest waveguide frequency of oscillations of the polydisperse chain is found from the equation $\det[AP(\lambda)] = 0$. For small τ, this frequency is calculated from the formula

$$\lambda_1(\xi, k, \tau) = \sqrt{2\tau(1 - \cos\xi)/[(k + \tau - k\tau)(k\tau - k\text{æ}^2 + \text{æ}^2)]}. \tag{3.1}$$

The right side of (3.1) coincides with the right side of the dispersion relation for the creeping mode (2.3).

For small values of τ and with other parameters being equal, depending on the concentration a minimum of the waveguide frequency, which is reached at the point $k = 0.5$, exists. For small τ, the lowest waveguide frequency of a polydisperse chain of bubbles is determined only by concentration. This means that, for $\tau \ll 1$, the polydispersity of a chain of bubbles has no significance for the creeping mode.

One can show that the polydispersity influences higher-order waveguide modes compared to the creeping mode. For a water–air chain, Fig. 7 shows the frequency of the second waveguide mode versus the sizes of the first water drop k_1 and the first air bubble k_2 at a fixed water concentration of $k = k_1 + k_3 = 0.9$. The polydispersity of the chain of heterogeneities affects greatly the frequencies of the second waveguide mode.

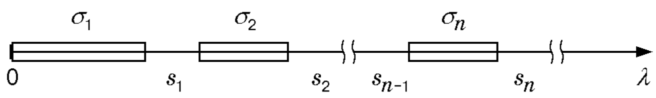

Fig. 8. Fine structure of the spectrum.

4. Resonance Properties of the Chain and the Spectrum Structure. The spectral properties of the problem T and the dispersion relations for the mono- and polydisperse chains of heterogeneities allow us to describe their resonance properties.

Fine Structure of the Spectrum. Dispersion relations (2.2) for a monodisperse chain and (3.1) for a polydisperse chain make it possible to determine the pass and stop bands of the problem of the propagation of acoustic waves through a one-dimensional-periodical chain of heterogeneities. It is important that, for mono- and polydisperse chains, a pass band that corresponds to the creeping mode, adjoins the zero (the pass band σ_1 in Fig. 8).

It is necessary to note that the number of pass and stop bands is infinite. The fine structure of the spectrum of the frequencies of a problem that describes the propagation of acoustic waves through a one-dimensional-periodical chain of heterogeneities is shown in Fig. 8 (σ_n and s_n are the pass and stop bands, respectively, where $n = 1, 2, \ldots$).

To study the propagation of wave trains and the resonance properties of one-dimensional-periodical chains of heterogeneities, it is necessary to examine the group velocity $C_g^{(n)}(\xi, k, \tau)$ of the waveguide modes $\lambda_n = \lambda_n(\xi)$ ($n = 1, 2, \ldots$). By virtue of the symmetry of the problem $T(\xi)$ with respect the wavenumber ξ at the points $\xi = m\pi$ ($m = 0, \pm 1, \pm 2, \ldots$), the equalities $C_g^{(n)}(0, k, \tau) = 0$ and $C_g^{(n)}(\pm\pi, k, \tau) = 0$ hold for corresponding values of the waveguide frequencies $\lambda_n(0)$ and $\lambda_n(\pm\pi)$, where $n = 1, 2, \ldots$. One should note that the values of the waveguide frequencies $\lambda_n(0)$ and $\lambda_n(\pm\pi)$ are the boundaries of the pass and stop bands; the values of $\lambda_n(0)$ belong to the stop bands s_n, and the values of $\lambda_n(\pm\pi)$ belong to the pass bands σ_n ($n = 1, 2, \ldots$).

Localized Resonance. Let a compact source with frequency λ_s be in a one-dimensional-periodical chain of penetrable heterogeneities. Because the group velocity is the propagation velocity of the energy of the waveguide mode, at frequencies of the source $\lambda_s \neq 0$ belonging to the boundaries of pass bands, resonance phenomena arise. In this case, at any moment of time the source energy is localized in its neighborhood. A similar phenomenon arises in the case where the oscillation frequency belongs to a certain stop band ρ_n ($n = 1, 2, \ldots$). The following theorem is true.

Theorem 1. *If the frequency of an oscillation source $\lambda_s \neq 0$ coincides with the boundary of a certain pass band or belongs to a certain stop band, then the source energy is localized in its neighborhood at each moment of time: the oscillation amplitude near the source increases as a function of time* (*resonance phenomena occur*).

For resonance values of the frequency of forced oscillations, concrete values of the amplitude are determined by the saddle-point method [10] with the use of (2.2).

Synchrophasotron Resonance. Let the oscillation sources $f(x, t)$ such that $f(x, t) = \exp(-i\omega_s t) f_1(x)_{x \in M_1}$, $f(x, t) = \exp(-i\omega_s t) f_2(x, t)_{x \in M_2}$, and

$$f(x + 1, t) \equiv \exp(i\xi) f(x, t) \equiv \exp(i\xi - i\omega_s t) f(x) \qquad (4.1)$$

be in a chain of penetrable heterogeneities. The acoustic pressure perturbation satisfies the equations of steady-state oscillations with dimensionless frequency $\lambda_s = L\omega_s/c_1$, which have the form

$$p_{xx}^{(1)} + \lambda_s^2 p^{(1)} = f_1(x), \qquad p_{xx}^{(2)} + \lambda_s^2 æ^2 p^{(2)} = f_2(x). \qquad (4.2)$$

Transmission conditions (1.2) should be satisfied at the boundaries of two media. Problem (4.2), (1.2) describes forced oscillations in a chain of heterogeneities; hereafter, it is called the problem $TF(\xi)$. Because the oscillation sources satisfy the phase-shift condition (4.1) at a certain value of ξ, the solution of the problem $TF(\xi)$ passes to the solution of the same problem under the action of any element of the translation group $\{T_1\}$ and satisfies condition (1.7). Let $\lambda^*(\xi)$ be a certain waveguide frequency of the problem $T(\xi)$. Since $T(\xi)$ is the homogeneous problem $TF(\xi)$, in the case where the dimensionless oscillation frequency of the source λ_s tends to a certain waveguide frequency $\lambda^*(\xi)$, a resonance occurs. The amplitude of the running wave increases unboundedly with time (as in synchrophasotrons). The following theorem is true.

Theorem 2. *If in a periodical chain of heterogeneities the oscillation sources with phase shift ξ are distributed periodically in adjacent fundamental domains of the translation group* (4.1)*, a resonance arises when the oscillation frequency of the source λ_s coincides with a certain waveguide frequency $\lambda^*(\xi)$. This resonance is called a synchrophasotron-type resonance.*

5. Conclusions. In calculating the wave propagation in inhomogeneous media, it is necessary to take into account the interaction between adjacent heterogeneities. This changes significantly the waveguide and resonance properties of an inhomogeneous medium and gives rise to the appearance of a series of low-frequency waveguide oscillations near the chain of heterogeneities. The waves that creep over the chain correspond to these oscillations. The low-frequency eigenoscillations caused by the interaction of adjacent heterogeneities should be taken into account in studying the propagation of the initial perturbation and the response of the structure to forced oscillations. The polydispersity of bubbles does not exert a great effect on the low-frequency waveguide oscillations of an inhomogeneous medium if the density ratio tends to zero (infinity). In this case, the lowest oscillation frequencies are determined by the linear concentration of bubbles, the period of the chain, and the phase shift of oscillations.

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